Chair of Experimental Solid State Physics, LMU Munich

“Introduction to Graphene and 2D Materials”

SS24 Lecture 7, 3/6/2024

Outline - Lecture 7

- Reminder about the Landau Level quantization and the Quantum Hall Effect.

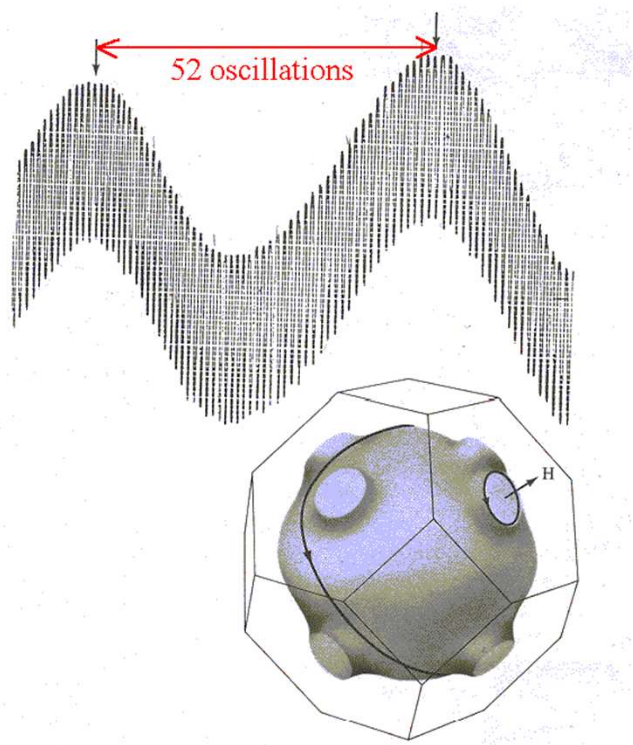
- Consequences of the Dirac equation:

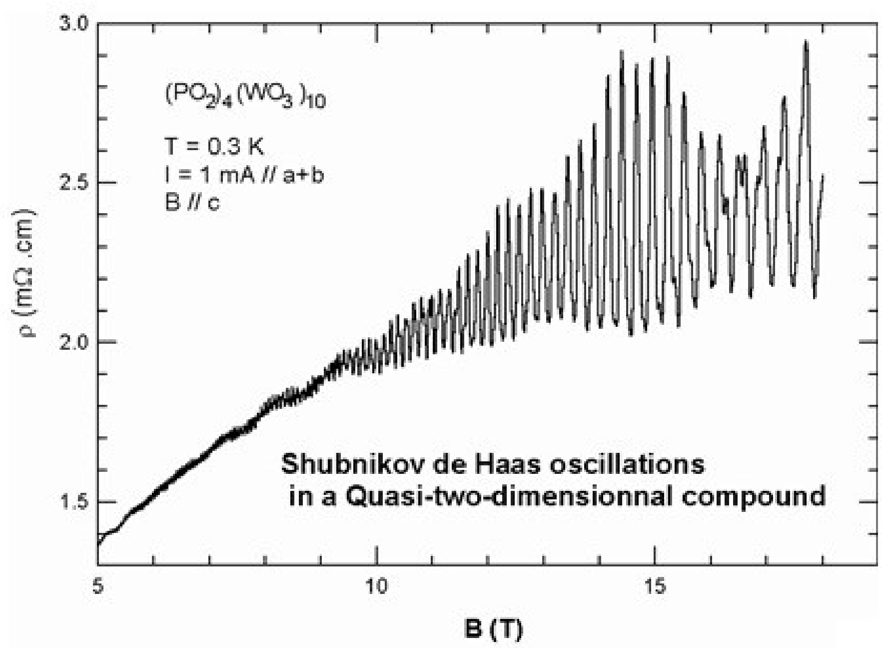
  - Relativistic Quantum Hall effect in graphene
  - Landau Fan diagram
  - Zeeman splitting and QH ferromagnetism
  - π-Berry's phase

Quantum oscillations

- Orbits in k-space are always in planes perpendicular to B.
- The electronic density of states at the Fermi energy E_F determines most of a metal's properties. Therefore there are many types of quantum oscillations with the magnetic field.
- Therefore the metal's properties (which depend on the energy level density of states at E_F) will oscillate as B changes, with a period given by:

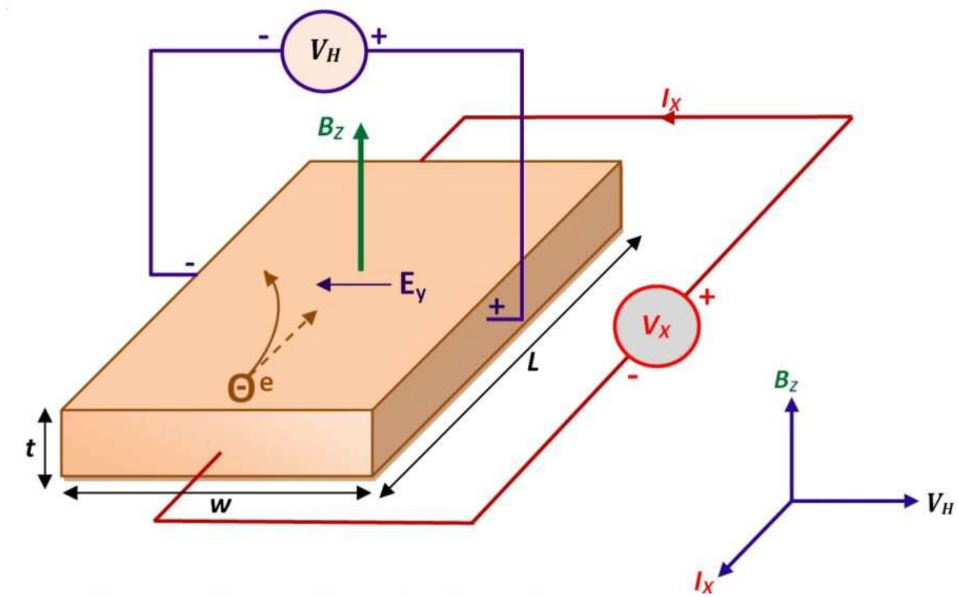
$$\Delta\left(\frac{1}{B}\right) = \frac{2\pi e B}{\hbar A}$$



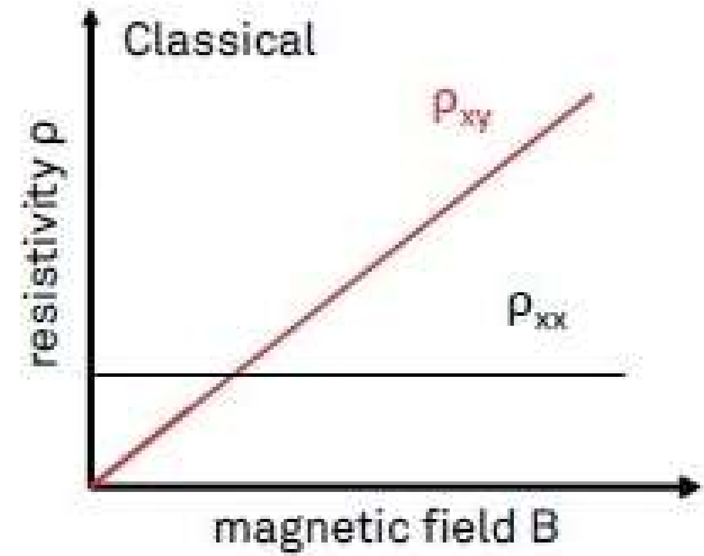


Large B-fields – Quantum Hall effect

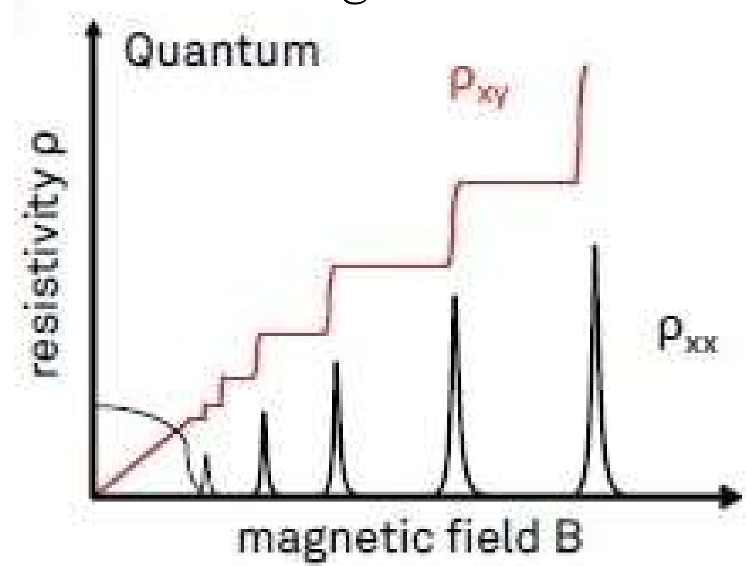
Hall effect measurement scheme:



Small B:



Large B:



Lorentz force is balanced by electric force: $e(v \times B) = eE$

Current: $I = neAv$ (n – carrier density, A - area, v – drift velocity)

Hall voltage: $V_H = Ew = IB/net$ ($t = 1$ in 2D)

Hall resistance: $R_{xy} = V_H/I = B/net$ ($t = 1$ in 2D)

Free electron in a B-field

$$H = \frac{\pi^2}{2m} = \frac{(\vec{p} - q\vec{A})^2}{2m}$$

where $\vec{A}$ is the vector potential that defines the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Choosing the Landau gauge $\vec{A} = B_o x\hat{y}$ for $\vec{B} = B_o\hat{z}$, we have

$$H = \frac{p^2}{2m} - \frac{qB_o p_y}{m}x + \frac{q^2 B_o^2}{2m}x^2$$

If the particles are constraint to move in the $x - y$ plane, the ansatz

$$\psi_{p_y} = e^{\frac{ip_y y}{\hbar}}\phi_{p_y}(x), \ p_y = \hbar k_y$$

Eigenstates in B-field in 2D

Define $\ell_B^2 \equiv \frac{\hbar}{qB}$, $\omega_c \equiv \frac{|qB_o|}{m}$ and complete the square

$$\frac{1}{2m}\left(p_x^2 + m^2\omega_c^2(x - k_y\ell_B^2)^2\right)\phi_{p_y} = E(p_y)\phi_{p_y}$$

This is a harmonic oscillator at $x = k_y\ell_B^2$ with energy levels

$$\boxed{E_n = \hbar\omega_c(n + \frac{1}{2})}$$

And the final wave function

$$\boxed{\psi_{n,p_y} = e^{ik_y y}H_n(x - k_y\ell_B^2)e^{-\frac{(x-k_y\ell_B^2)^2}{4\ell_B^2}}}$$

where H_n are the Hermite polynomials. The energy levels (6) are called *Landau levels*. There are many quantum states for every Landau level i.e. for a given n, every p_y corresponds to a state with the same energy E_n.

Landau levels in a free electron picture

Eigenstates:

$$\psi_{n,p_y} = e^{ik_y y} H_n(x - k_y \ell_B^2) e^{-\frac{(x-k_y\ell_B^2)^2}{4\ell_B^2}}$$

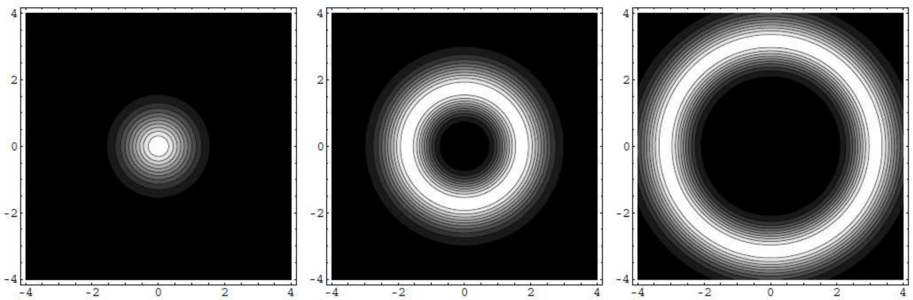
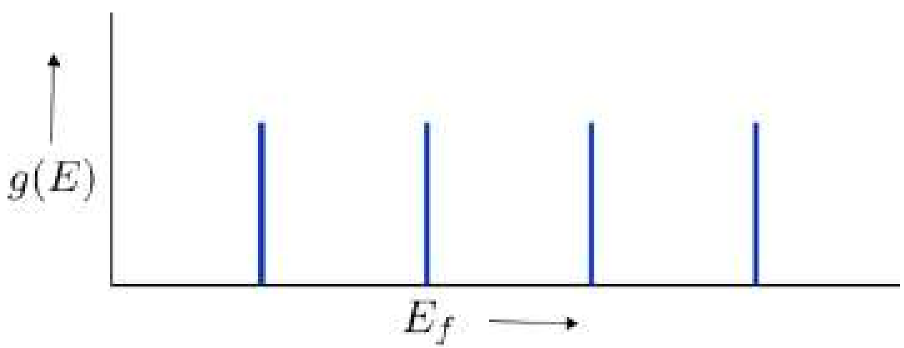


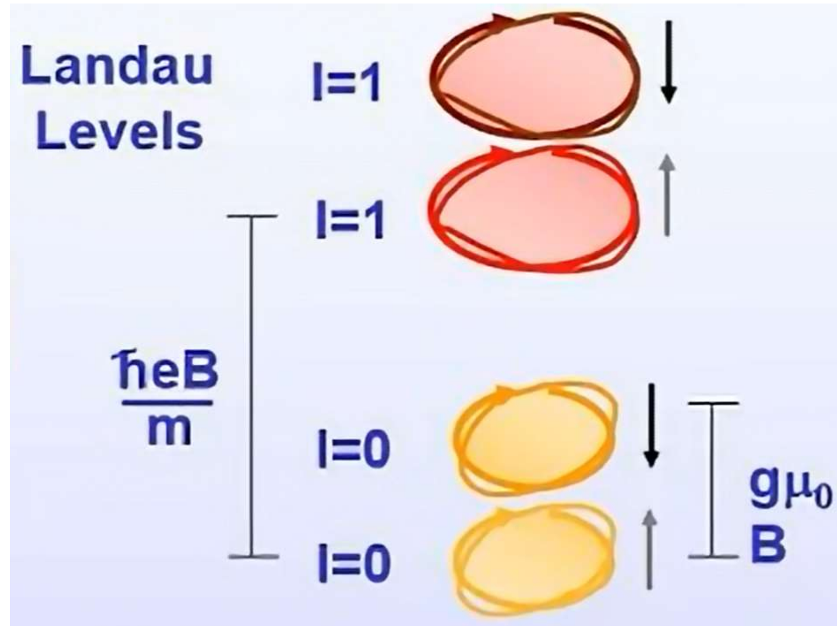
Figure 3: The ground state wave functions with $n = 0$, 3, and 10.

Eigenenergies:

$$E_n = \hbar\omega_c(n + \frac{1}{2})$$



Landau quantized states:



flux quantum: $\Phi_0 = h/e$

magnetic length: $l = r/\sqrt{n} = \hbar/eB$

cyclotron frequency: $\omega_C = eB/m$

Number of states

Suppose the system is of size $L_x \times L_y$, then the separation between harmonic oscillators

$$\Delta x = \Delta k_y \ell_B^2 = (\frac{2\pi}{L_y})\ell_B^2$$

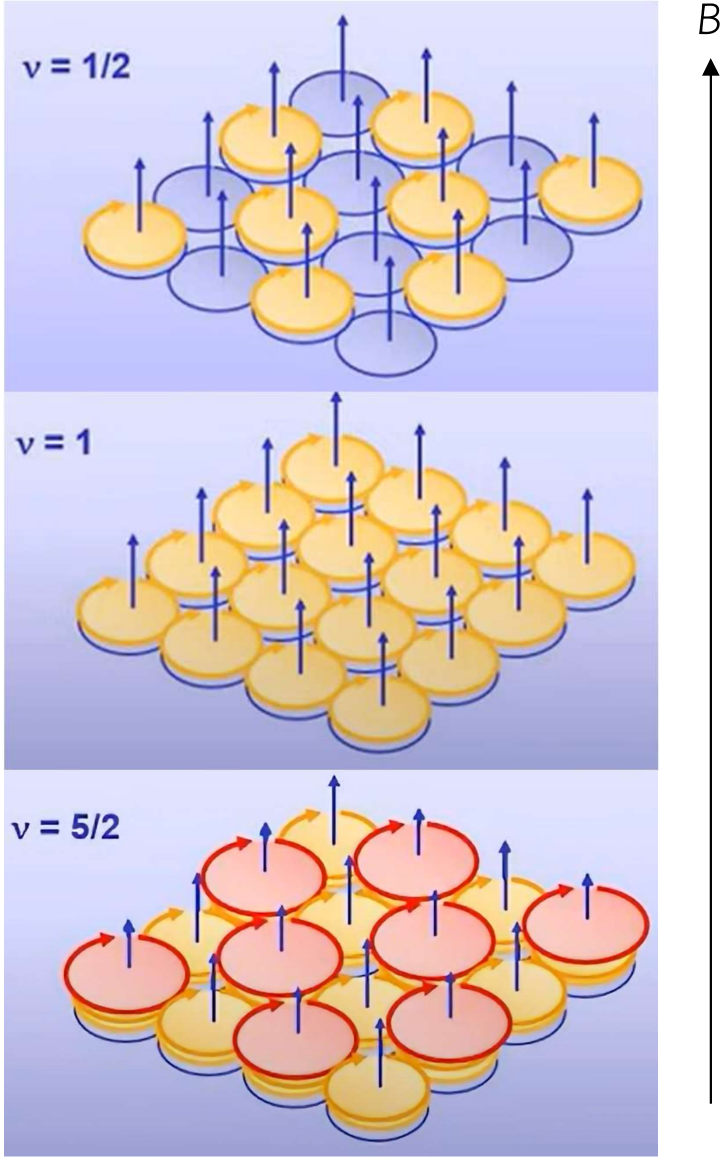
Thus the number of oscillators we can fit into the system

$$\boxed{N = \frac{L_x}{\Delta x} = \frac{L_x L_y}{2\pi \ell_B^2}}$$

Plugging in $\ell_B^2 \equiv \frac{\hbar}{qB}$ we see that for electrons

$$N = \frac{q}{\hbar} B L_x L_y = \frac{B L_x L_y}{\hbar / e} = \frac{\phi}{\phi_o}$$

Filling LLs in B-field



- LL orbitals become smaller with B, but bigger with n:

$$r = \sqrt{n\hbar/eB}$$

- Energy of the LLs increases with B and n:

$$E_n = \hbar\omega_c(n+1/2) = (n+1/2)\,\hbar eB/m$$

- Energy spacing between LLs increases with B:

$$E_n - E_{n-1} = \hbar\omega_c = \hbar eB/m$$

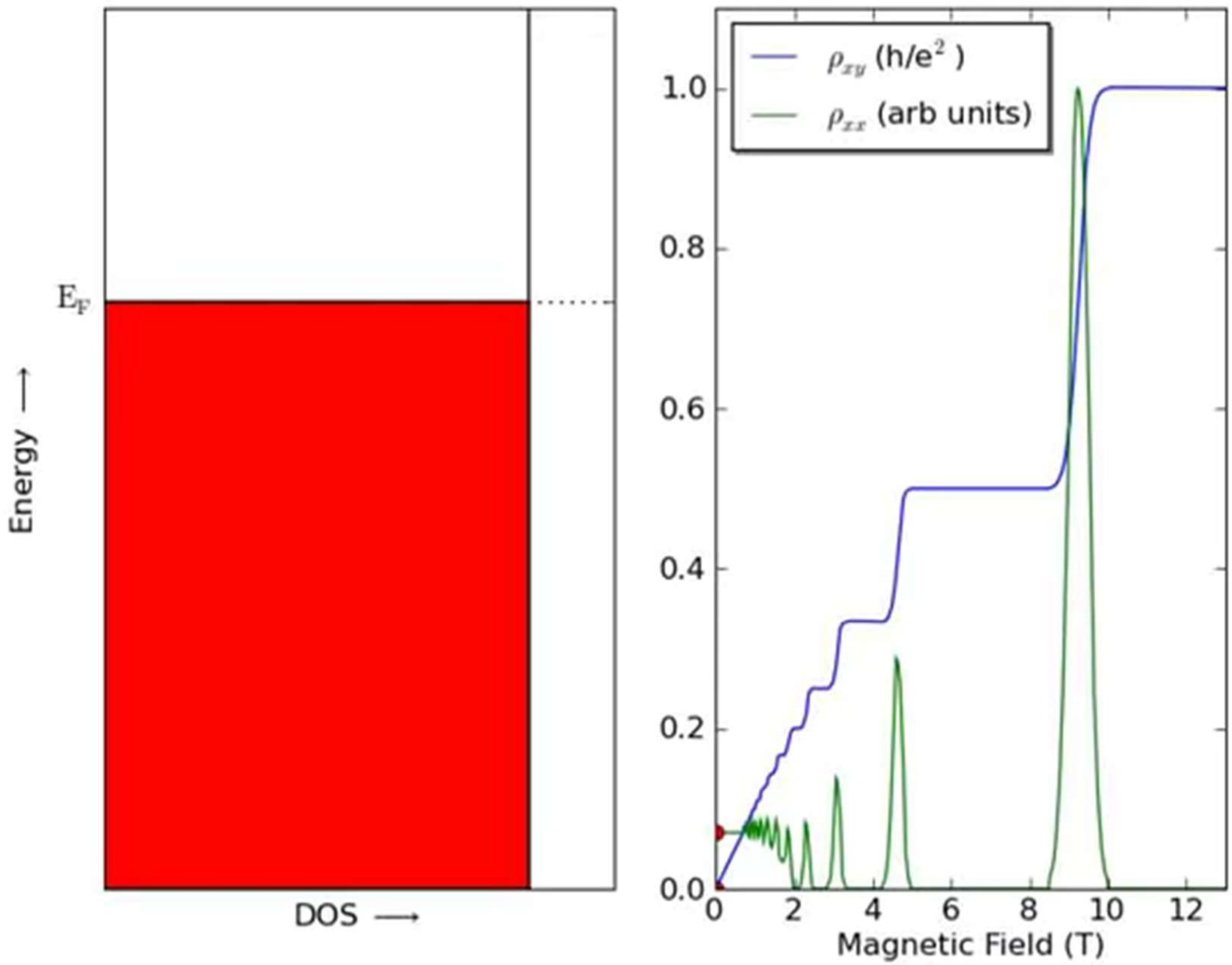
- Each Landau level holds the exactly same amount of states (electrons), where total number of states in each LL grows with B (g_s = 2 accounts for spin):

$$N = g_s L_x L_y/2\pi l_B^2 = g_s AB/\Phi_0 = g_s\Phi/\Phi_0$$

- filling factor = number of occupied LLs (below Fermi energy) - total number of electrons n_s devided by number of electrons in a LL (not accountig for degeneracy):

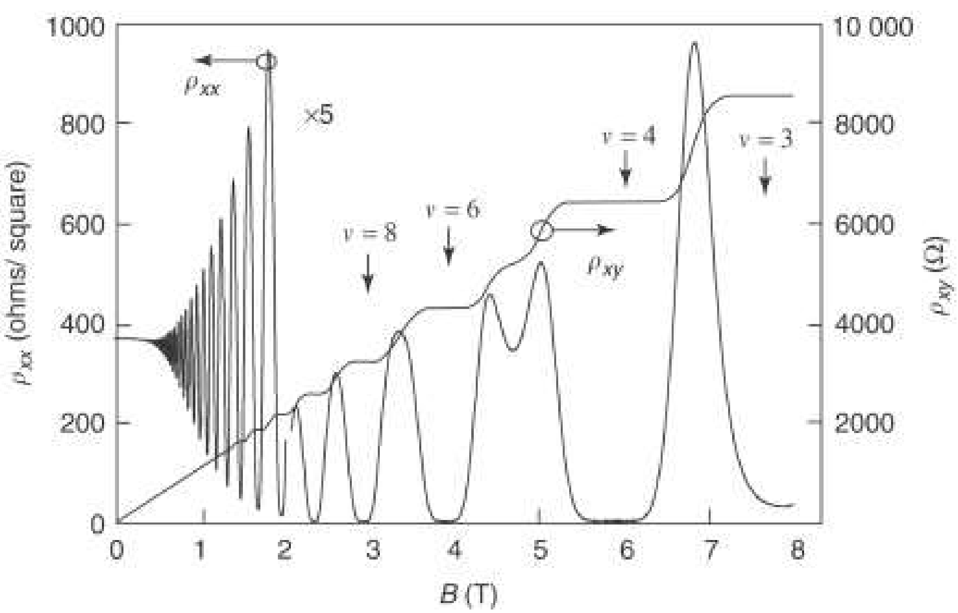
$$\nu = hn_s/eB$$

Linking filling of the LLs with transport measurements

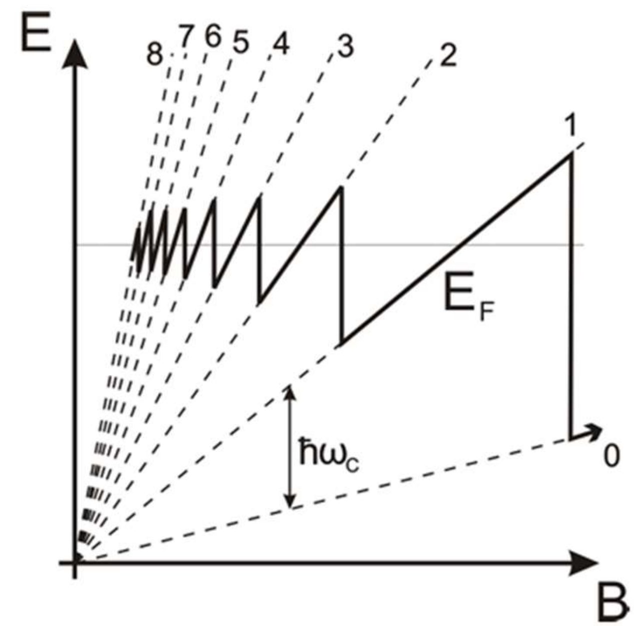


Shubnikov de Haas oscillations and Quantum Hall Effect

Shubnikov de Haas oscillations:



Depopulation of the LLs in B-field:



Number of filled (degenerate) LLs: $\frac{n_s}{N_L} = \frac{hn_s}{eB} \cdot \frac{1}{g_s g_v}$

Therefore, two consecutive minima obey the expression:

$$\Delta\left(\frac{1}{B}\right) = \frac{1}{B_{(i+1)}} - \frac{1}{B_{(i)}} = g_s g_v \cdot \frac{e}{hn_s} \tag{2.21}$$

In essence, the Shubnikov-de Haas minima are periodic in $\frac{1}{B}$. Using 2.21, one is also able to make a statement about the charge carrier concentration n_s:

$$n_s = g_s g_v \cdot \frac{e}{h}\left(\frac{1}{B_{(i+1)}} - \frac{1}{B_i}\right)^{-1} \tag{2.22}$$

Integer Quantum Hall effect

Early day 2D electron gas (2DEG) device:

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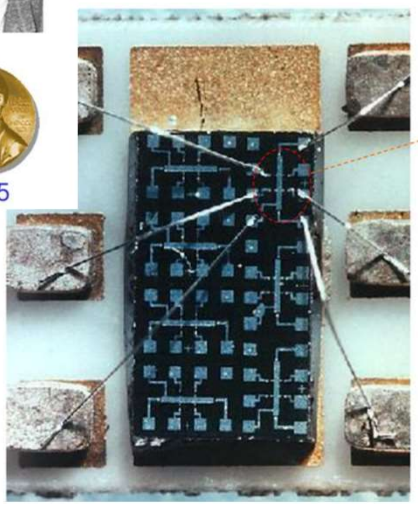
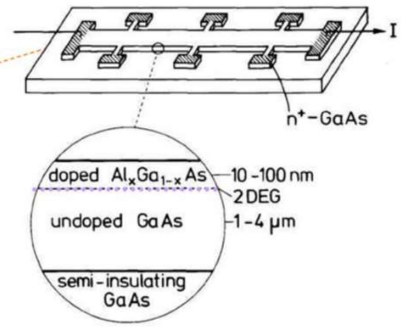
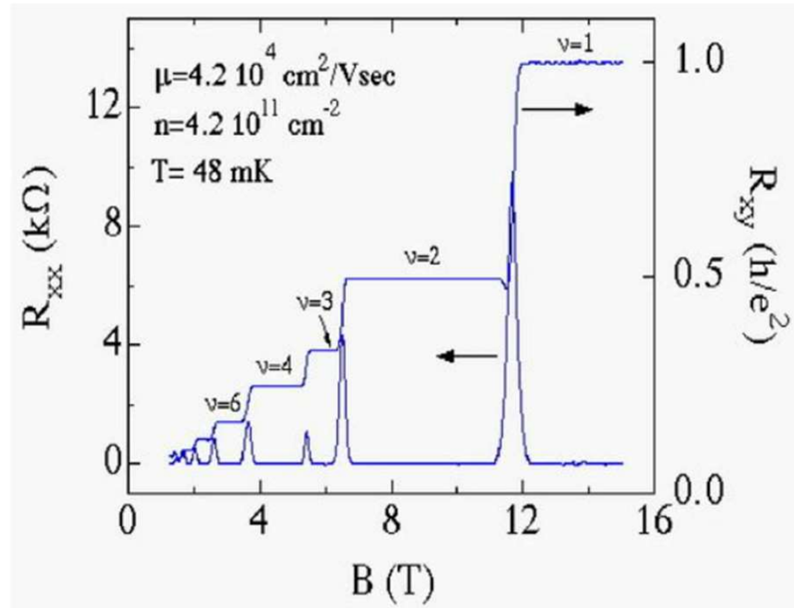




2-dim electron gas

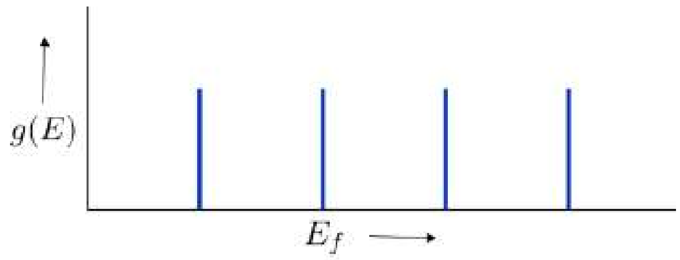
QHE in a 2DEG device:

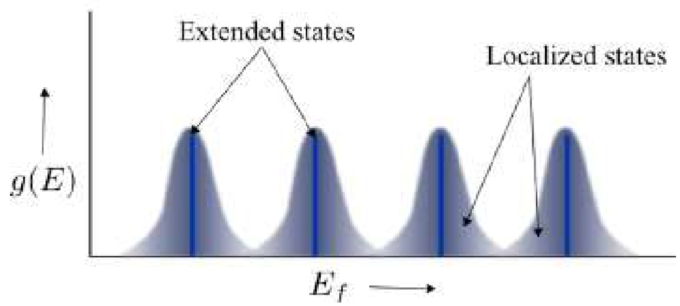


- Sharply quantized R_{xy} plateaus to units of h/e^2 – with a precision better than 1ppm.
- Vanishing R_{xx} in the same regions where R_{xy} quantized.
- Effect independent of shape/size of the sample.
- Observed in many different material platforms (Si MOSFET, GaAs, graphene, ZnO)

Disorder driven localization and delocalization

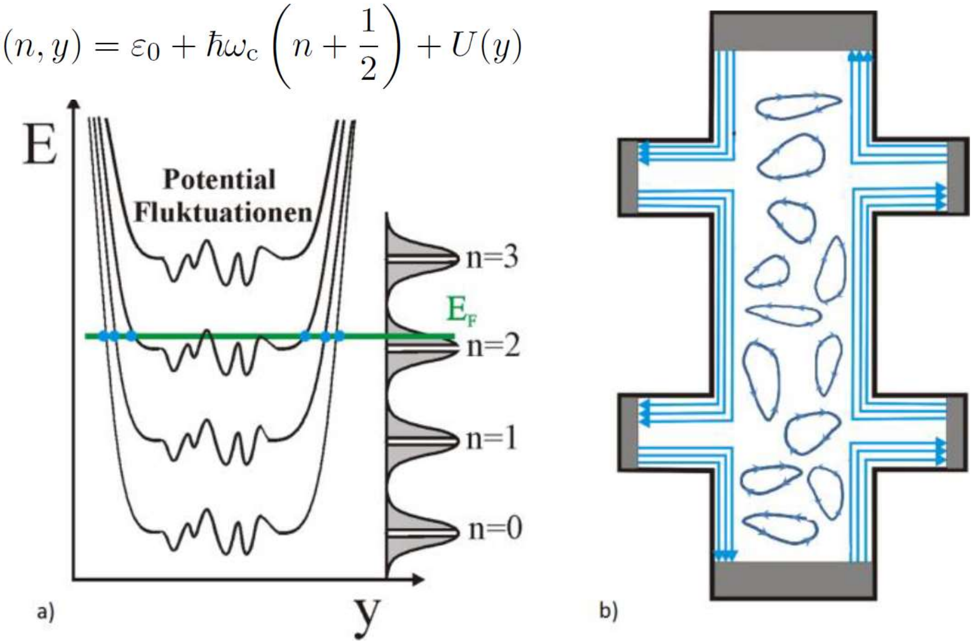
Disordered broadens LLs:



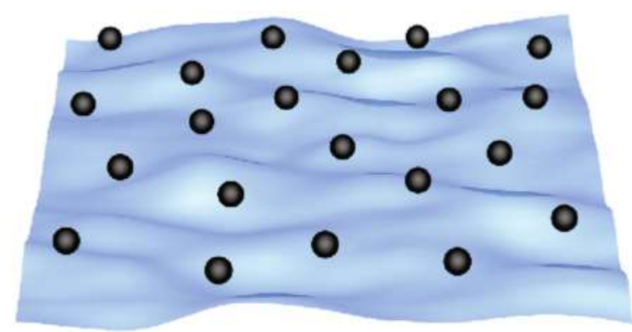


LL crossection in a realistic sample:

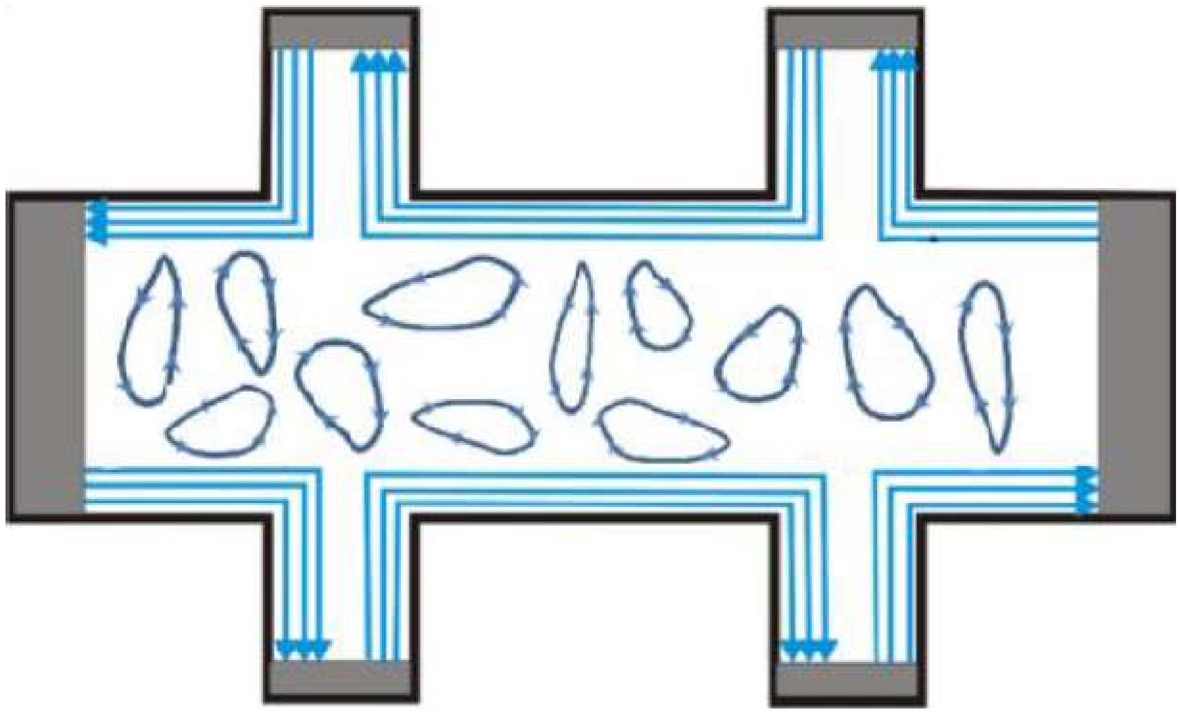
$$E(n,y) = \varepsilon_0 + \hbar\omega_c\left(n + \frac{1}{2}\right) + U(y)$$



- Disorder broadens LLs, so forming two types of states, localized orbital states in the bulk, and dissipationless edge states, that cannot scatter backwards.
- Confining potential forces LLs to fold upwards at the edges, and cross the Fermi energy, so forming conducting states at the edges with a linear dispersion → these give rise to plateaus.



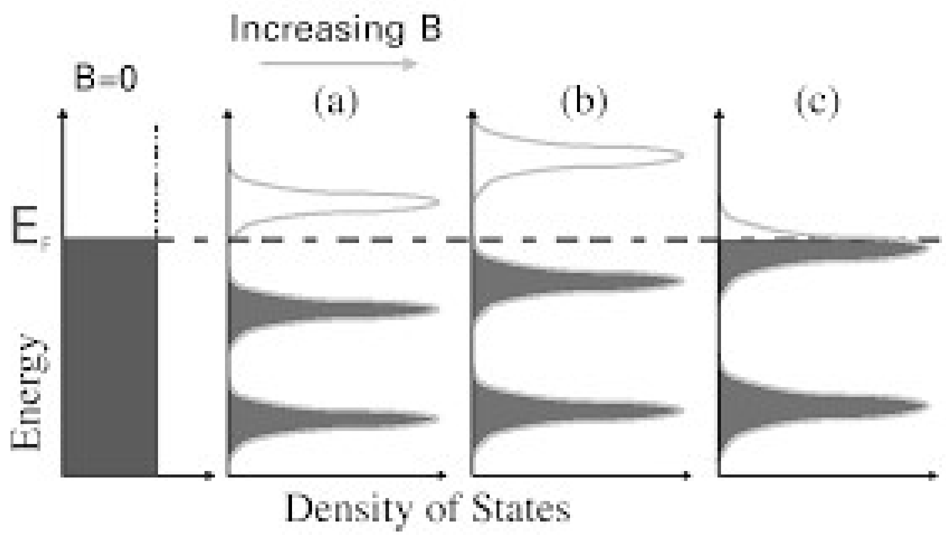
QHE – delocalized chiral 1D edge states



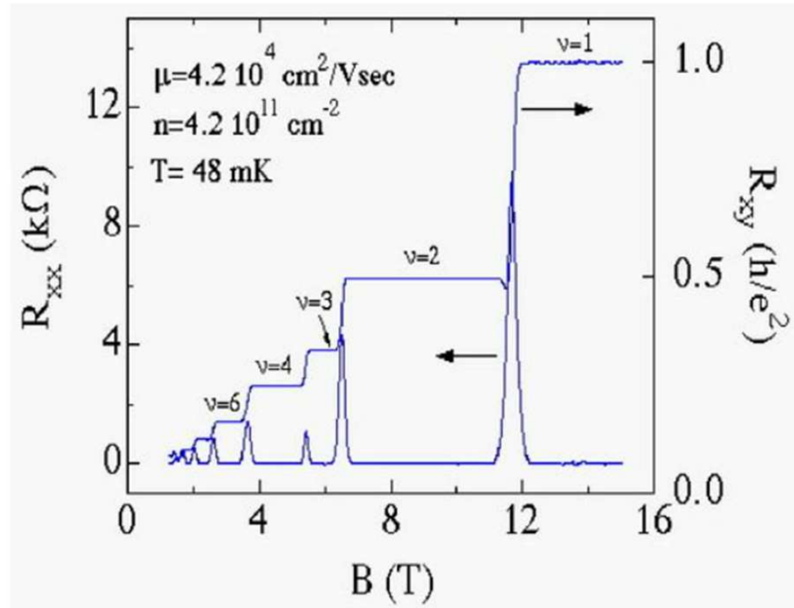
- Formation of chiral 1D edge states at the edges of the device.
- These states represent a novel order and ground states of matter.
- They are topologically protected and their exact quantization $R_{xy} = (h/e^2)/\nu$ follows from this protection (here $\nu = 3$).
- Number of edge states = Chern number (here C = +3, where + is clockwise and – is counterclockwise motion)

Integer Quantum Hall effect

LL and DOS evolution in B-field:



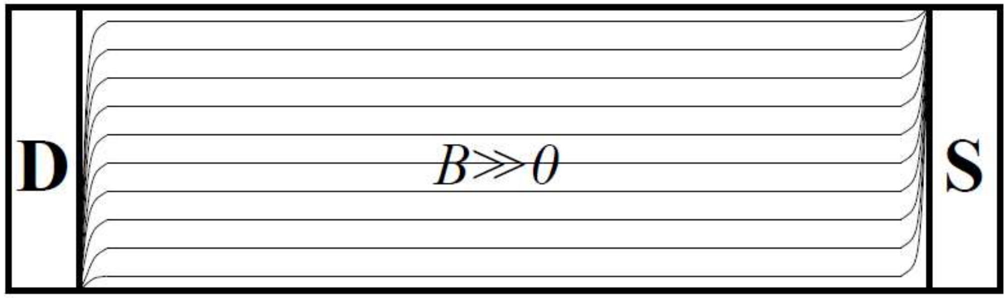
QHE in a 2DEG device:

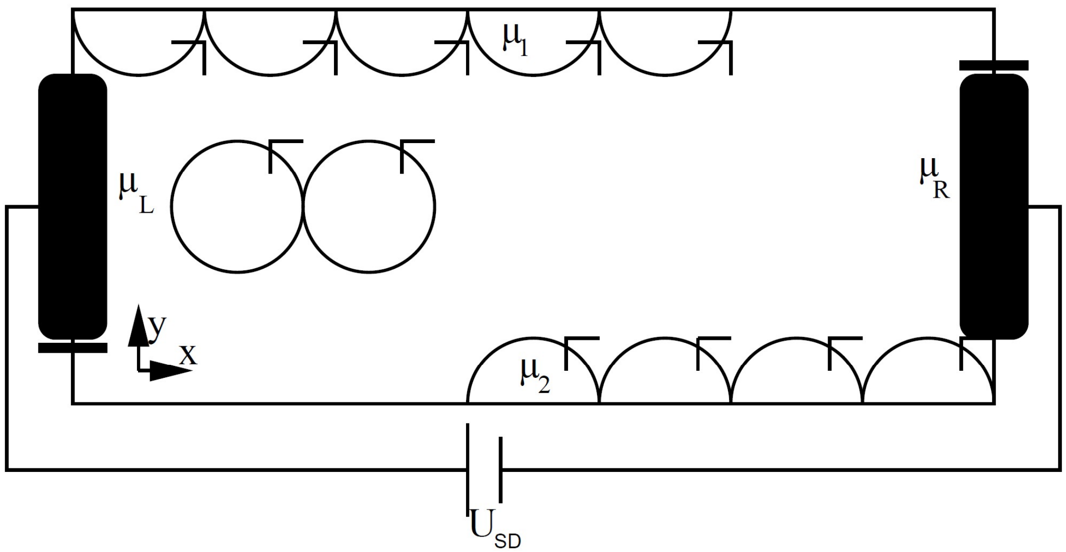


- Insulating-like state: Quantized R_{xy} plateaus and vanishing R_{xx} appear when E_F is inbetween two LLs.
- Increasing B-field spreads the entire LL spectrum, allowing for LL to continuously move through E_F .
- R_{xy} plateaus are quantized to the resistance quantum $R_{xy} = (h/e^2)/n,$ where n is an integer defined by the number of occupied LLs. An ideal 1D conduction channel carries this resistance.

→ However non of this yet can explain why such exactly quantized R_{xy} plateaus are formed, and why R_{xx} is vanishing.

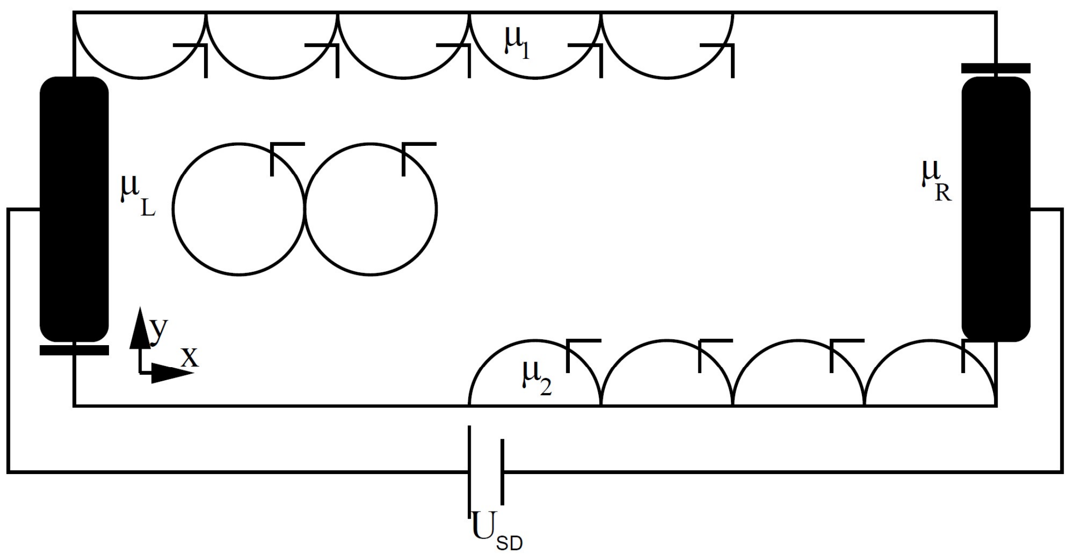
Vanishing Rxx of edge states





- Because the edge states move on a constant potential along most of the edge (except at the very contact), and also are protected from backscattering, the longitudinal resistance of these Rxx = 0 → they are almost dissipationless.

Quantization of Rxy



From the classical consideration presented earlier, we have already seen from the equipotential lines in Fig.2, that in a strong magnetic field the Hall-voltage is identical to the source-drain voltage ($U_{\rm H} = U_{\rm SD}$). When the edge channels are solely responsible for charge transport, this result is trivial. Because the edge channel is resistance-free, and therefore there is no voltage drop across the channel, i.e. in Fig. 7 $\mu_1 = \mu_{\rm L}$ and $\mu_2 = \mu_{\rm R}$, and the electrons in the upper channel (μ_1) move to the right, and in the lower channel (μ_2) to the left. The entire potential drop occurs only across a very small region, known as the 'hot-spots' (marked with thick lines in Fig. 7). The Hall voltage is then

$$U_{\rm H} \equiv U_{yx} = -\frac{1}{e}(\mu_1 - \mu_2) = -\frac{1}{e}(\mu_{\rm L} - \mu_{\rm R}) = U_{\rm SD}$$

Quantization of Rxy

In the following, we will derive the current carried by an edge channel and determine the conductance quanta. In general, the current carried by a charge Q is $I = \langle Q/t \rangle = \langle Q \rangle \langle 1/t \rangle$. If there are β electrons in an edge channel, then $\langle Q \rangle = -e\beta$. In accordance with the Pauli principal, there cannot be more than one electron having the same energy in a particular location. This extent of this region is given by the de-Broglie wavelength $\lambda = 2\pi/k_{\mathrm{F}} = h/mv_{\mathrm{F}}$. Therefore, the number of electrons that fit in the edge channel of length l is given by $\beta = l/\lambda = lmv_{\mathrm{F}}/h$ (or double as many when spin degeneracy is included). To determine the value of $\langle 1/t \rangle = \langle v \rangle / l$, we consider the electron velocity along both the edges;

$$\left\langle \frac{1}{t} \right\rangle = \frac{1}{l} \langle v_{\mathrm{LR}} - v_{\mathrm{RL}} \rangle$$

The relationship between $eU_{\mathrm{SD}} = \mu_{\mathrm{L}} - \mu_{\mathrm{R}}$ and the difference of the edge channe velocities is shown in Fig.10, and is given by

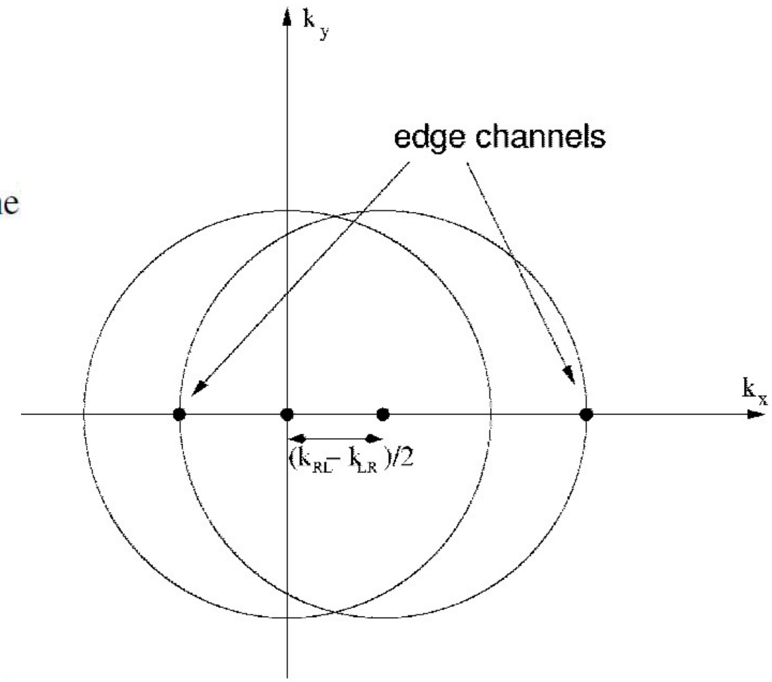
$$\begin{aligned} \mu_{\mathrm{L}} - \mu_{\mathrm{R}} &= \frac{1}{2} m \langle v_{\mathrm{LR}}^2 - v_{\mathrm{RL}}^2 \rangle \\ &= \frac{1}{2} m \langle (v_{\mathrm{LR}} + v_{\mathrm{RL}})(v_{\mathrm{LR}} - v_{\mathrm{RL}}) \rangle \\ &= \frac{1}{2} m 2 v_{\mathrm{F}} \langle v_{\mathrm{LR}} - v_{\mathrm{RL}} \rangle \end{aligned}$$

and with Eqn.(13), the current through an edge channel is then

$$I = \langle Q \rangle \left\langle \frac{1}{t} \right\rangle = -\frac{e}{h}(\mu_{\mathrm{L}} - \mu_{\mathrm{R}}) = \frac{e^2}{h} U_{\mathrm{SD}} = \frac{e^2}{h} U_{\mathrm{H}} \qquad (14)$$

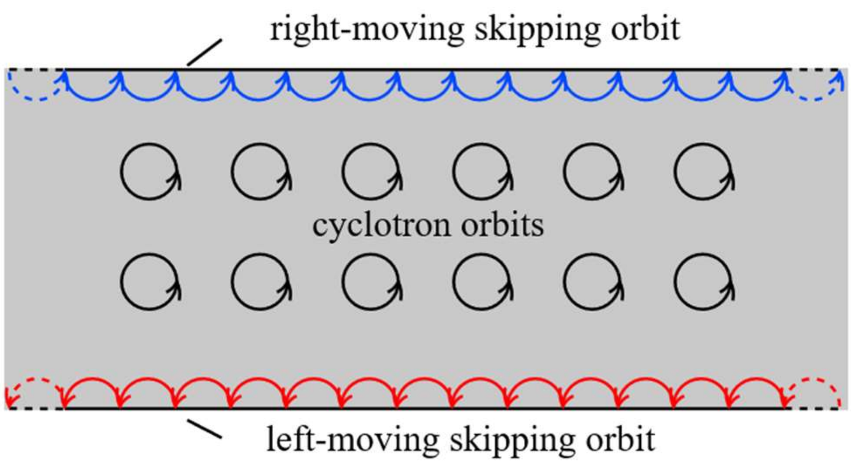
The transverse resistance per edge channel is therefore

$$R_{xy}^{\mathrm{Kanal}} = \frac{U_H}{I} = \frac{h}{e^2}$$

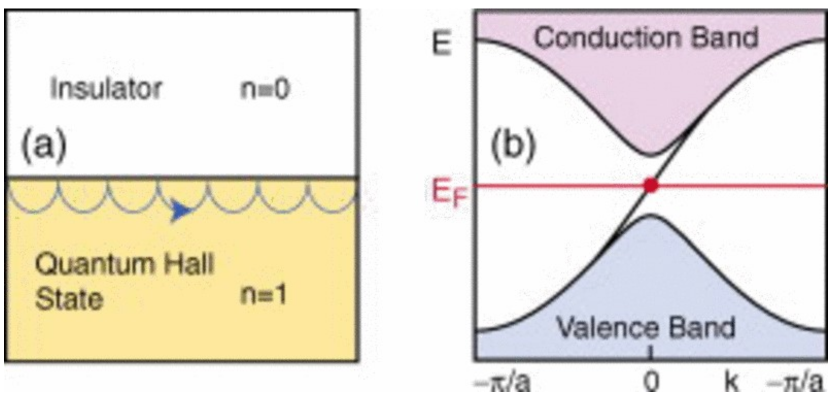


Topologically protected edge and localized bulk states

Schematic of a Quantum Hall State:



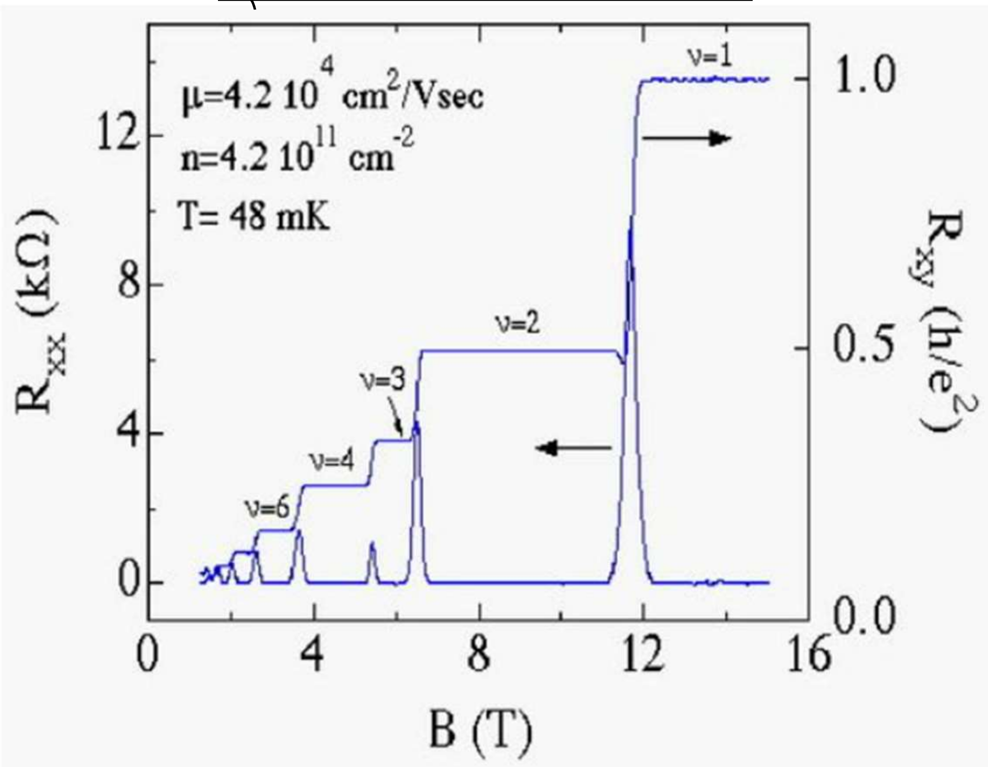
Band-diagram of edge states:



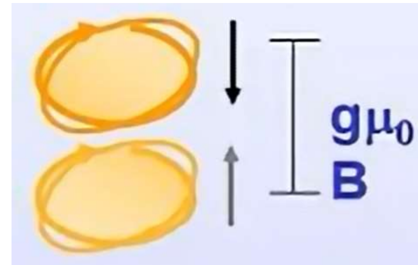
- Orbital states in the bulk are localized → bulk is insulating and a mobility gap is formed (Anderson localization).
- 1D edge states moving in one direction are formed at the edge → these are topologically protected, as back-scattering is not allowed, resulting in perfectly quantized and dissipation-less states.
- Symmetry protected topological states → a topological invariant protects these states and their quantization.

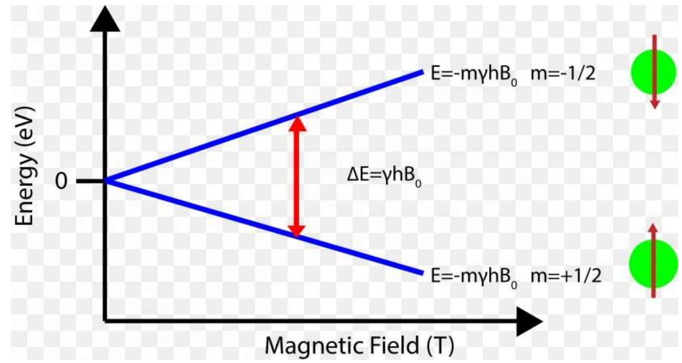
Analyzing the exact QHE – Zeeman splitting of LLs

QHE in a 2DEG device:



Zeemann splitting of the spins:





- R_{xy} plateaus are quantized to the resistance quantum $R_{xy} = (h/e^2)/n$, where n is an integer defined by the number of occupied LLs.
- Each Landau level holds the exactly same amount of states (electrons), where total number of states in each LL grows with B (g_s = 2 accounts for spin):
- $N = g_s L_x L_y / 2\pi l_B^2 = g_s AB/\Phi_0 = g_s \Phi/\Phi_0$
- filling factor = number of occupied LLs (below Fermi energy) - total number of electrons n_s devided by number of electrons in a LL (not accountig for degeneracy): $\nu = hn_s/eB$

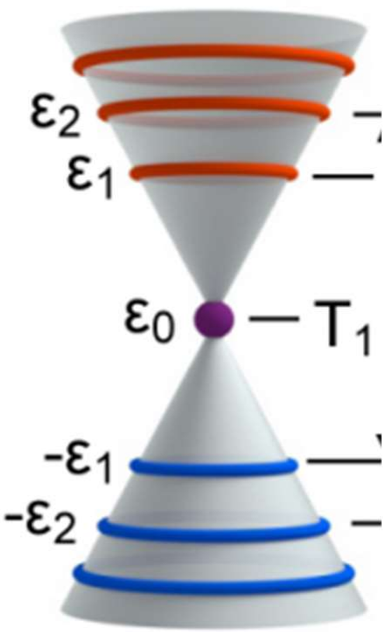
LLs in graphene

Dirac Hamiltonian in B-field:

$$H = v\vec{\sigma}\cdot(\vec{p} + e\vec{A})$$

Schroedingers equation:

$$\hbar v \begin{pmatrix} 0 & k_x - \partial_y + \frac{eB}{\hbar}y \\ k_x + \partial_y + \frac{eB}{\hbar}y & 0 \end{pmatrix} \phi(y) = E\phi(y)$$



Ansatz for the wavefunction:

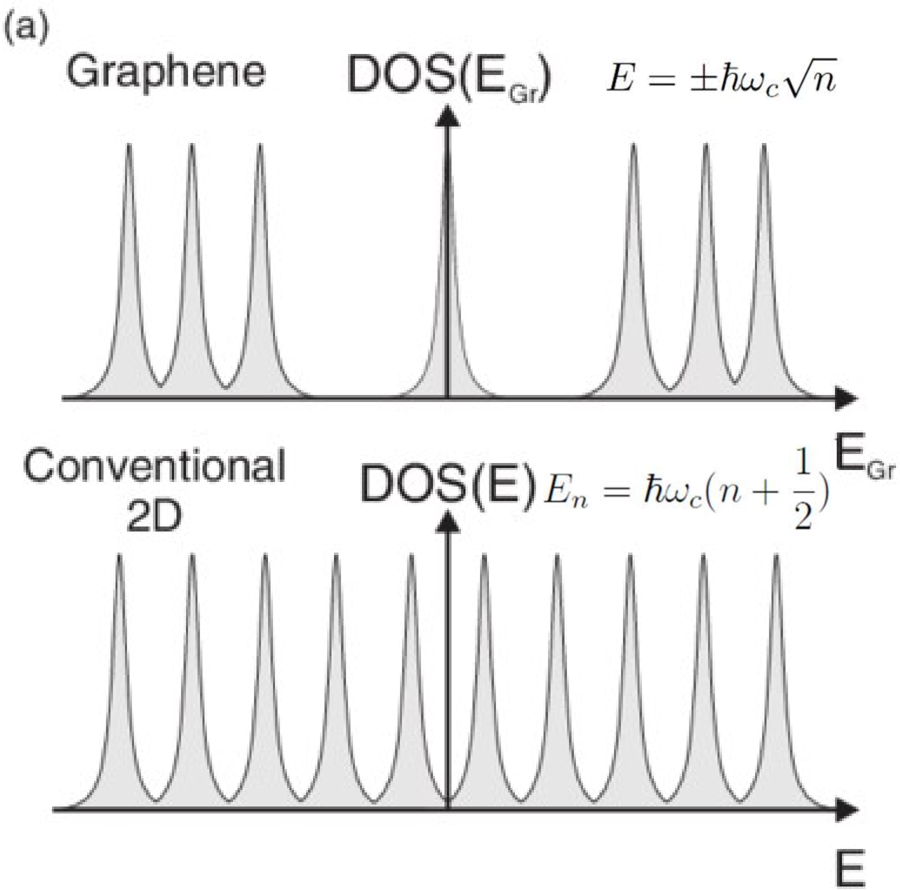
$$\psi(x,y) = e^{ik_x x}\phi(y)$$

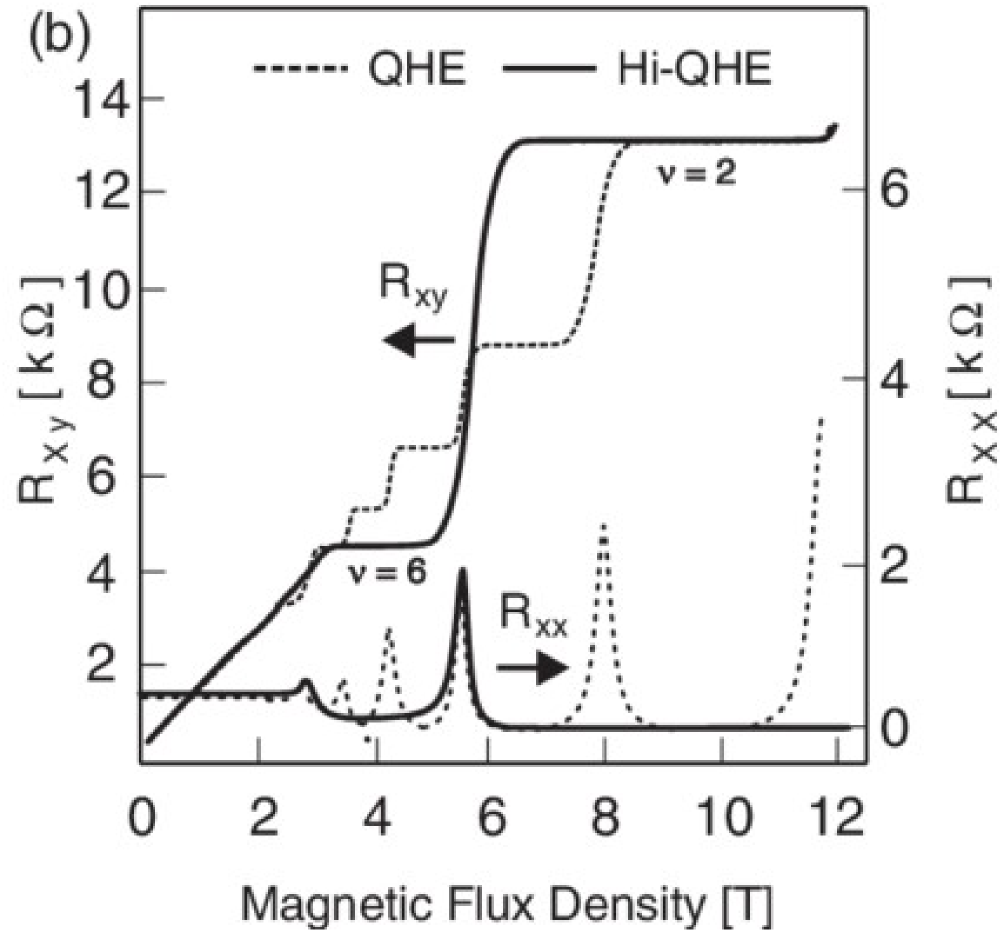
Eigenenergies:

$$E = \pm\hbar\omega_c\sqrt{n}$$

$$\ell_B^2 \equiv \frac{\hbar}{qB}, \; \omega_c \equiv \frac{|qB_o|}{m}$$

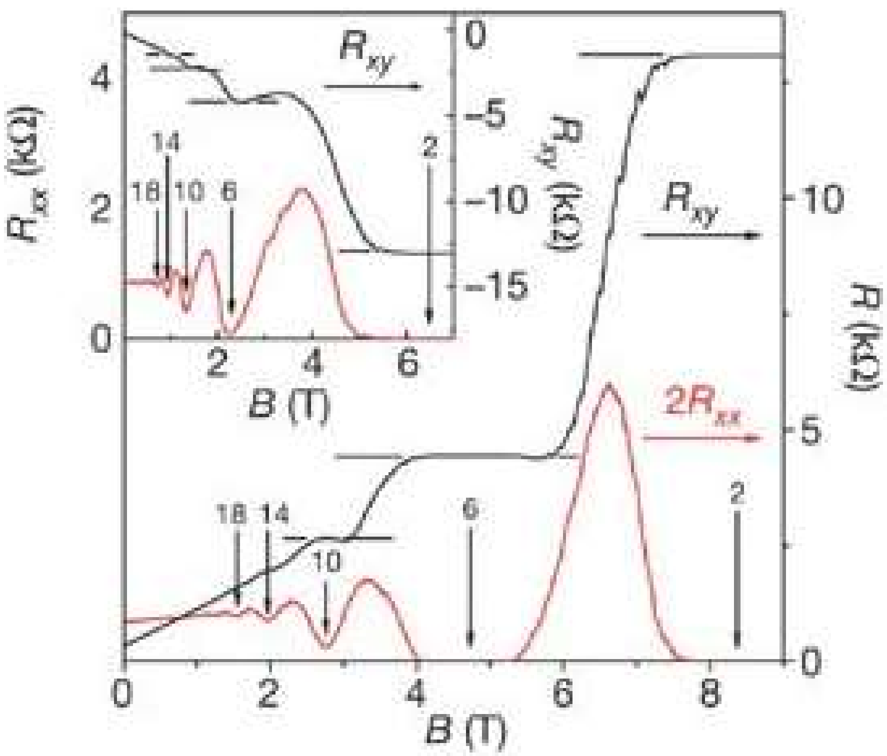
Quantum Hall effect in graphene

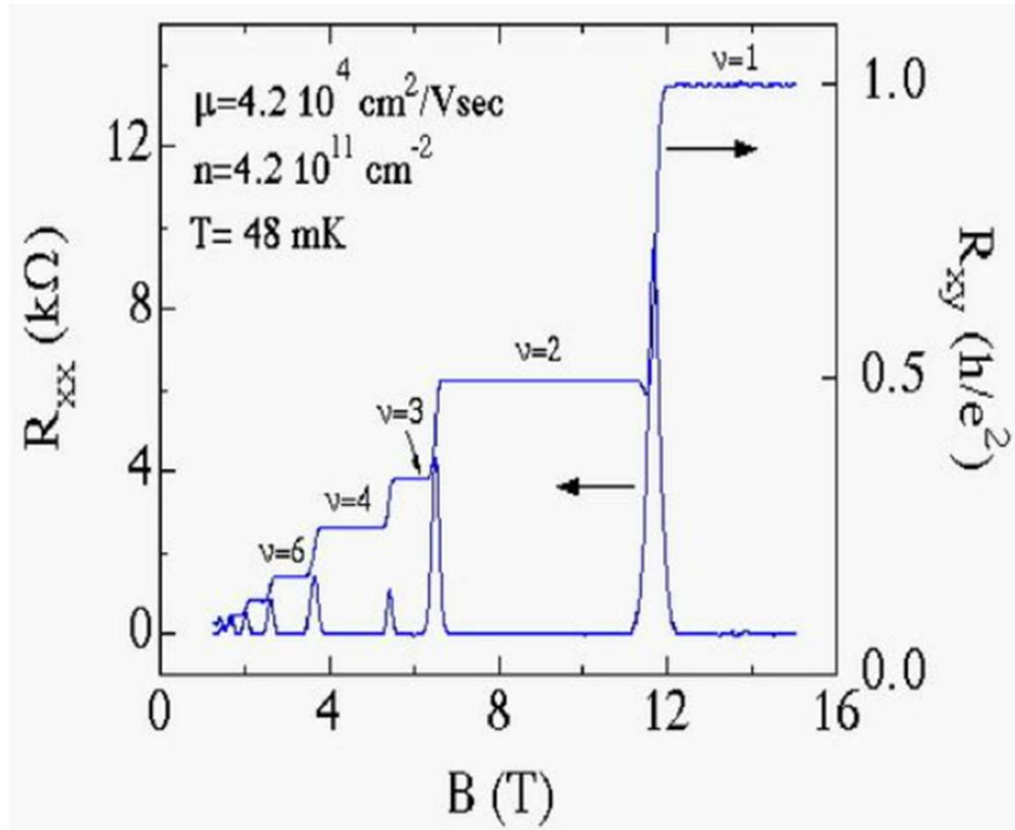




- Clearly an integer QHE with R_{xy} = $(h/e^2)/n$, where n is an integer.
- However, the sequence of LLs is quite different, where R_{xy} = $(h/e^2)/n$ takes values n = 2, 6, 10 etc.
- This implies a degeneracy of 4 (spin+valley), and a zero-energy LL, which is not present in normal 2DEGs.

Quantum Hall effect in graphene





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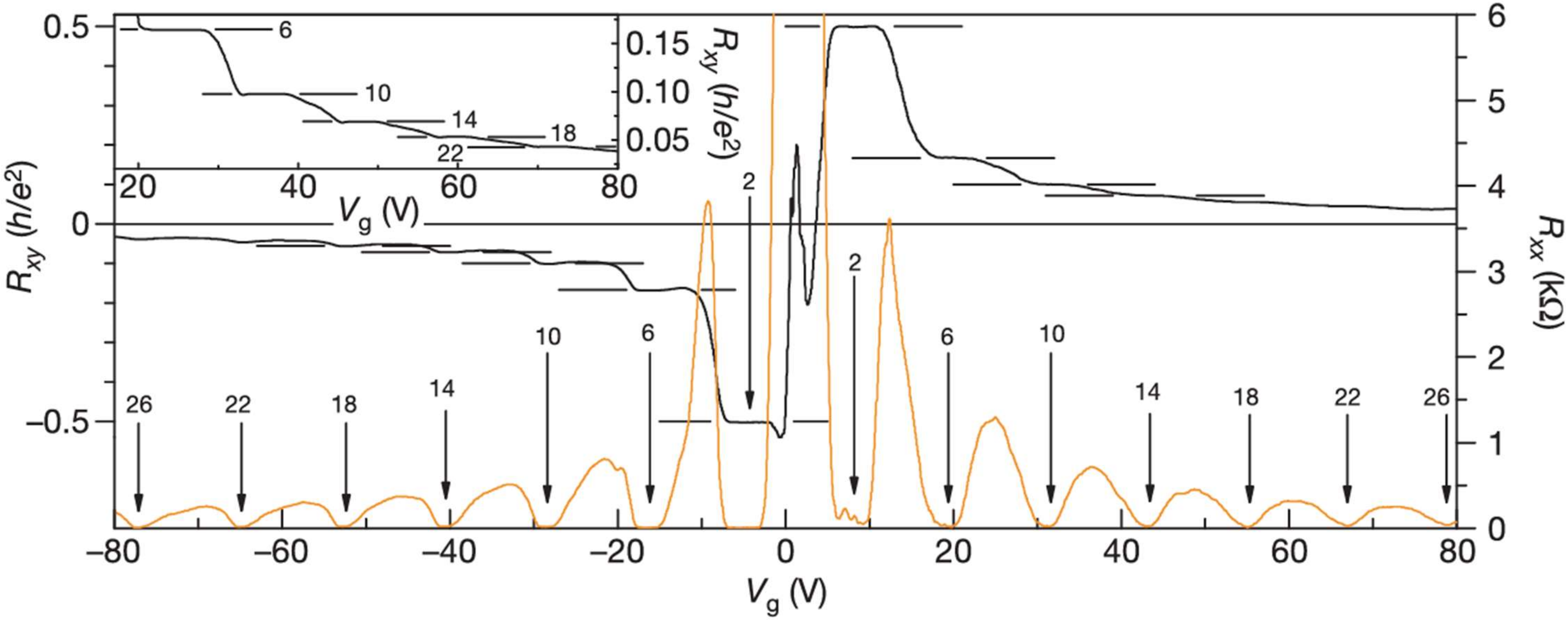
Gate control of carrier density

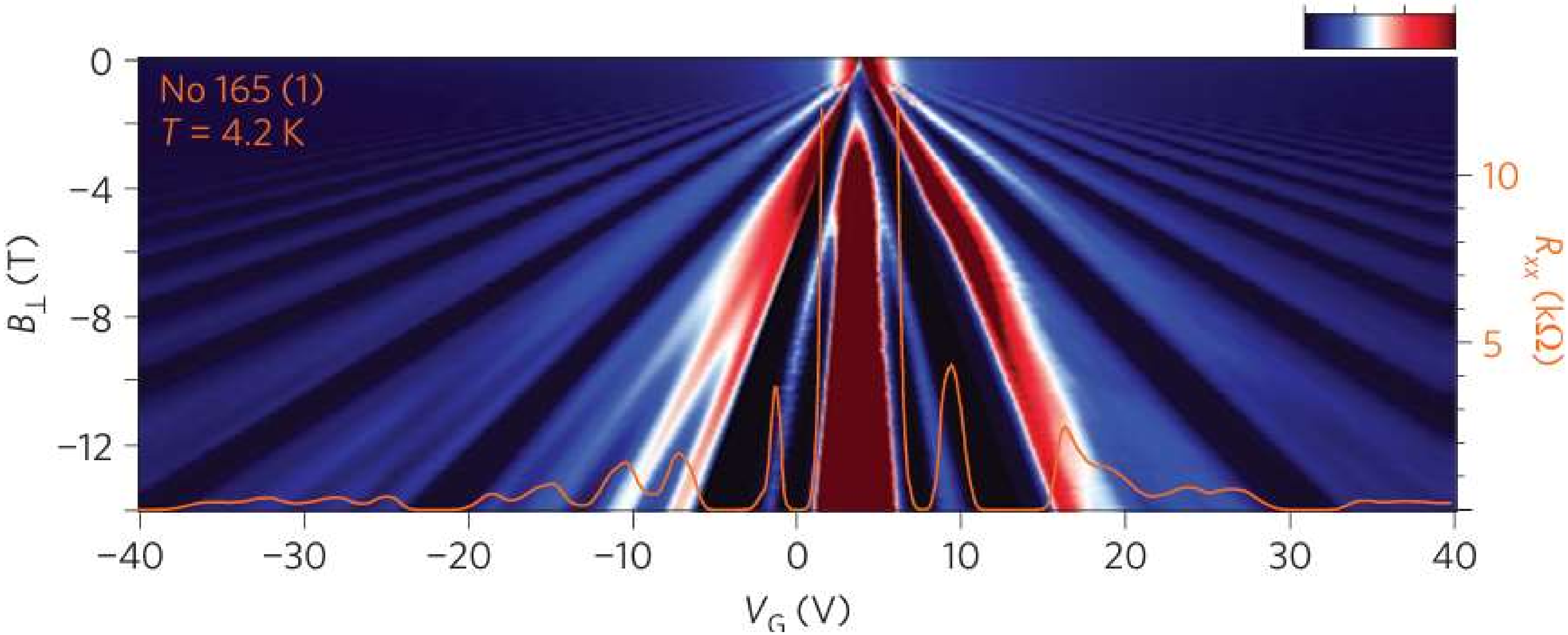


- Energy of the LLs increases with B and n: $E_n = \hbar\omega_c\sqrt{n} = \hbar eB\sqrt{n}/m$

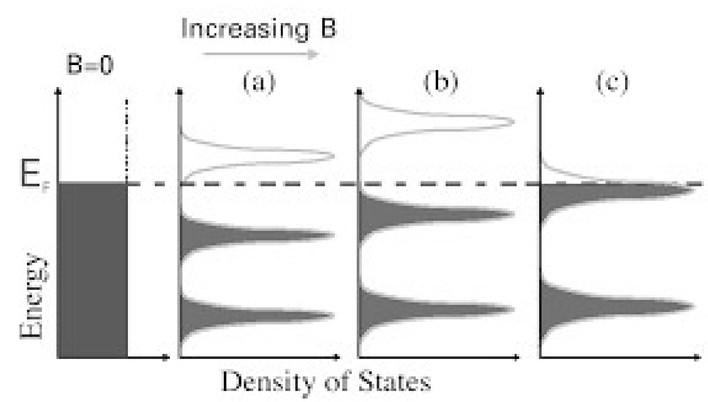
- Each Landau level holds the exactly same amount of states (electrons), where total number of states in each LL grows with B (g_s = 2 accounts for spin, g_v =2 accounts for valley):
- $N = g_s g_v L_x L_y / 2\pi l_B^2 = g_s g_v AB/\Phi_0 = g_s g_v \Phi/\Phi_0$

- filling factor = number of occupied LLs (below Fermi energy) - total number of electrons n_s devided by number of electrons in a LL (not accountig for degeneracy):
- $\nu = hn_s/eB$

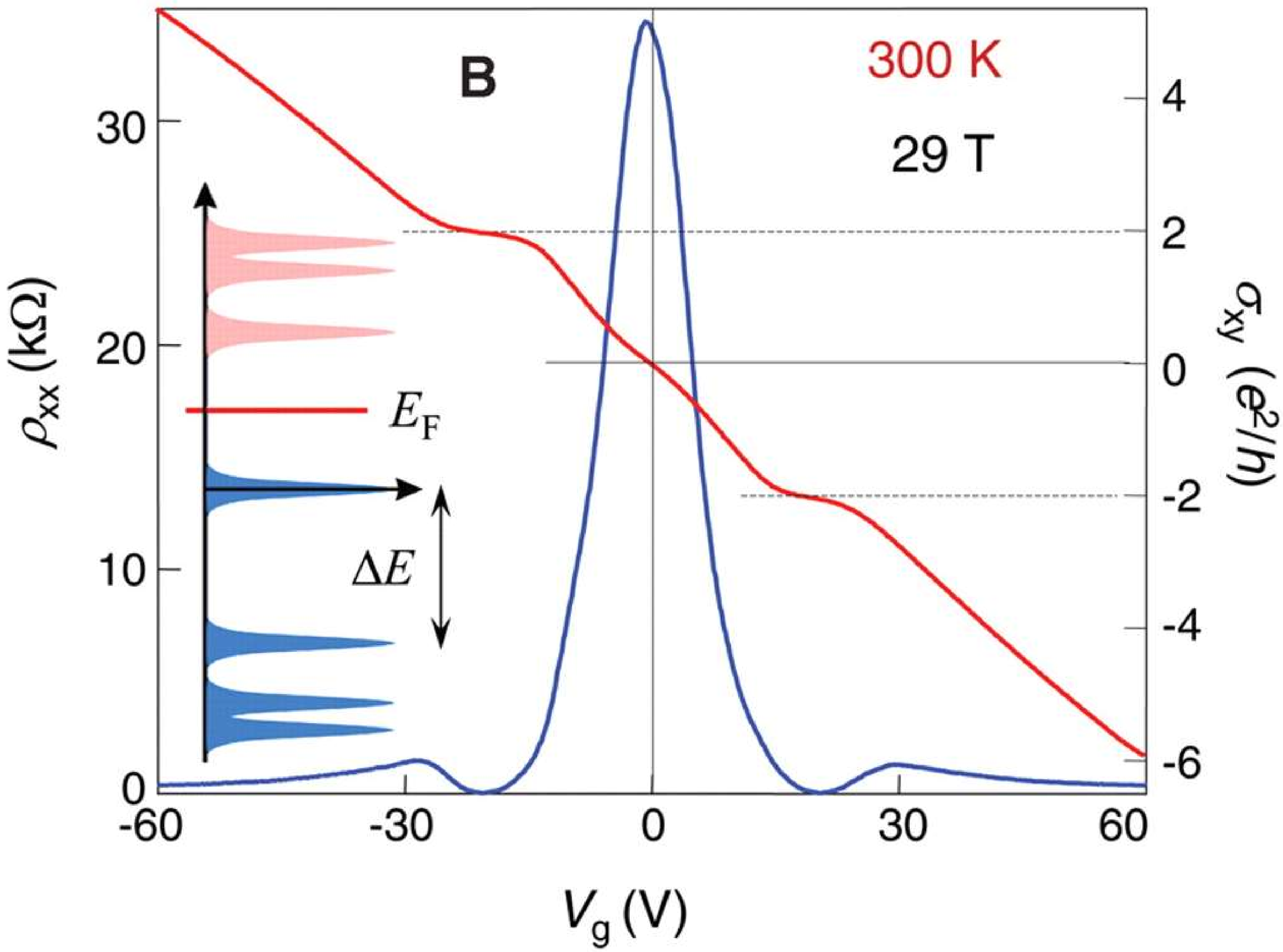
Landau fan in the n vs. B phase space



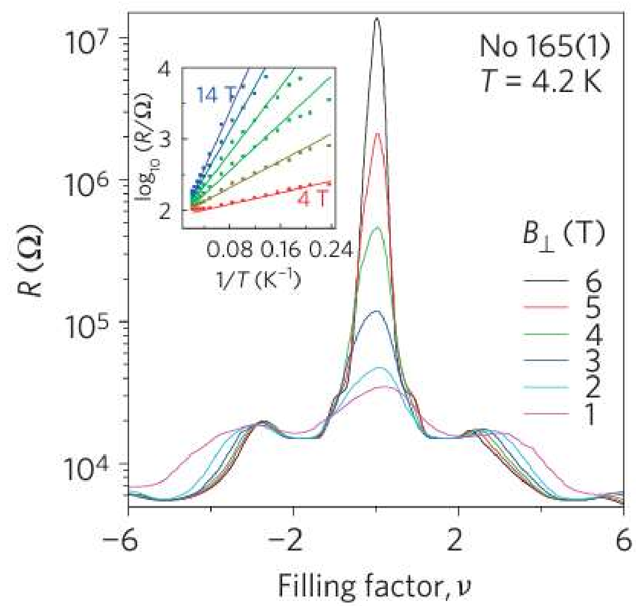
- There is a linear dependence of the number of states in one of the LLs N vs. B:
- $N = g_s g_v L_x L_y / 2\pi l_B^2 = g_s g_v AB/\Phi_0 = g_s g_v \Phi/\Phi_0$
- For a fixed filling factor ν there is a linear dependence of the carrier density in one of the LLs N vs. B:
- $n_s(\nu) = \frac{\nu e B}{h}$

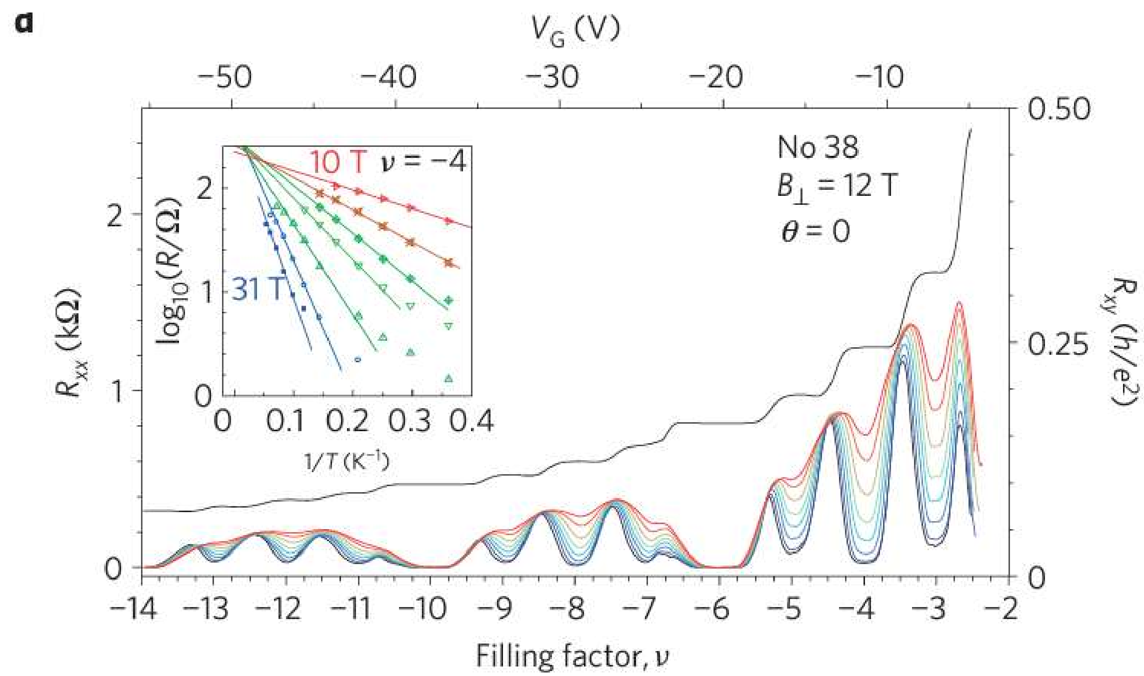


Room temperature Quantum Hall effect in graphene



Spin and valley splitting at large B





- Each Landau level holds the exactly same amount of states (electrons), where total number of states in each LL grows with B (g_s = 2 accounts for spin, g_v =2 accounts for valley):
- $N = g_s g_v L_x L_y / 2\pi l_B^2 = g_s g_v AB/\Phi_0 = g_s g_v \Phi/\Phi_0$

- filling factor = number of occupied LLs (below Fermi energy) - total number of electrons n_s devided by number of electrons in a LL (not accountig for degeneracy):
- $\nu = h n_s / eB$

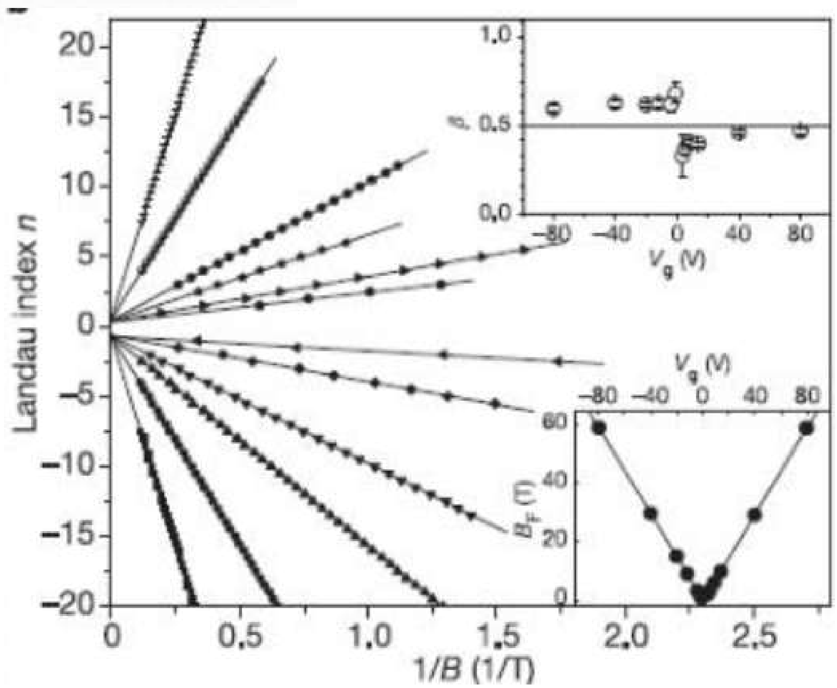
- Spin and valley degeneracies are lifted at large B-field due to Zeeman splitting and electron-electron interactions.

Zero electron mass and Berry curvature in graphene

- The location of 1/B for the nth minimum (maximum) of Rxx, counting from B=BF, plotted against n(n +1/2)
- Slope (lower inset) =BF
- Intercept (upper inset) = Berry's phase

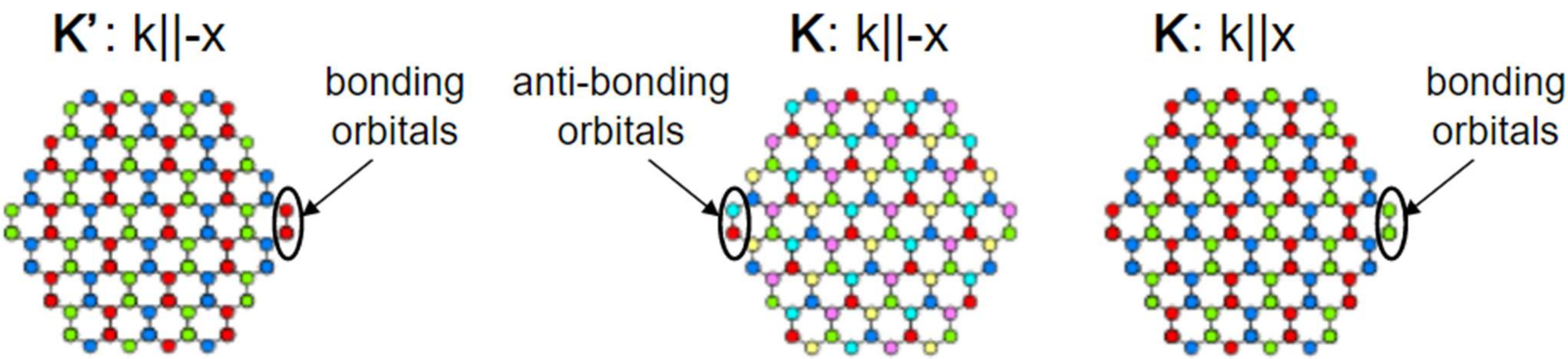
$$\Delta R_{xx} = R(B,T)\cos[2\pi(B_F/B + 1/2 + \beta)]$$

- B_F = Shubnikov-de Haas Oscillation Frequency in 1/B
- β = Berry Phase
- Aquired when quasiparticle moves between sublattices

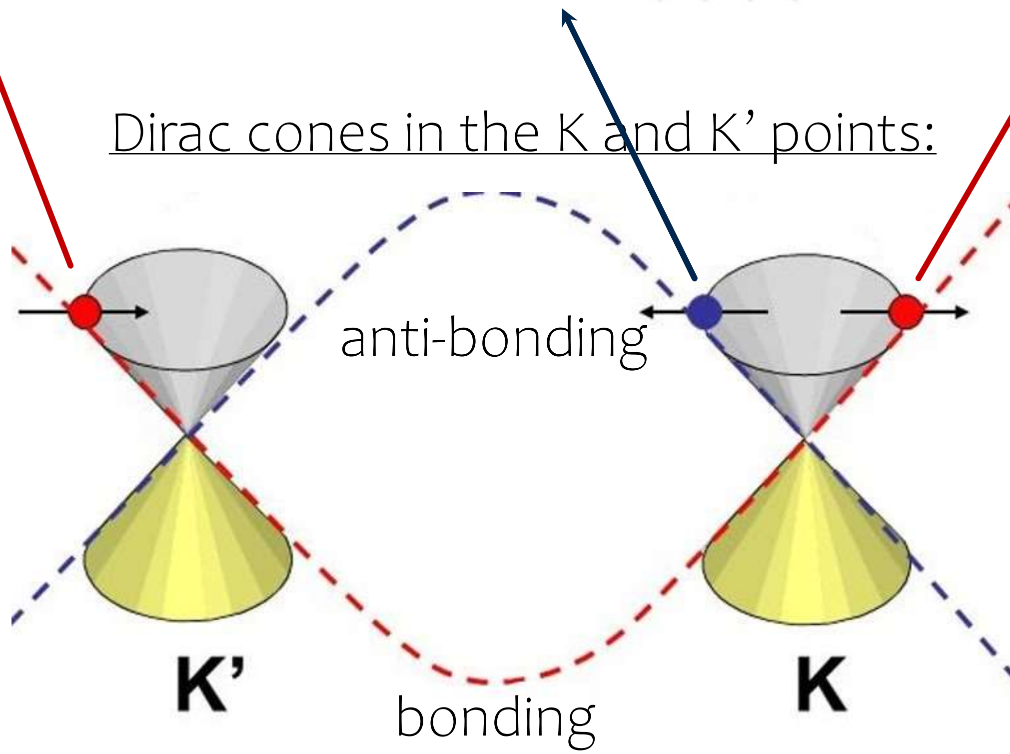


Real-space wave-functions and pseudo-spin texture

Real space wave-functions:

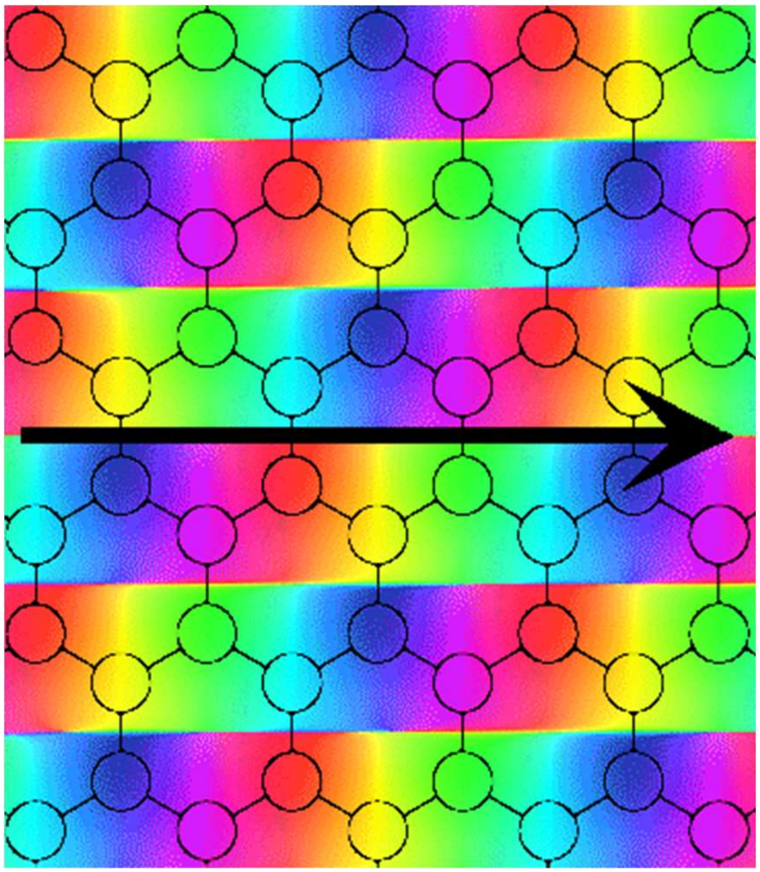


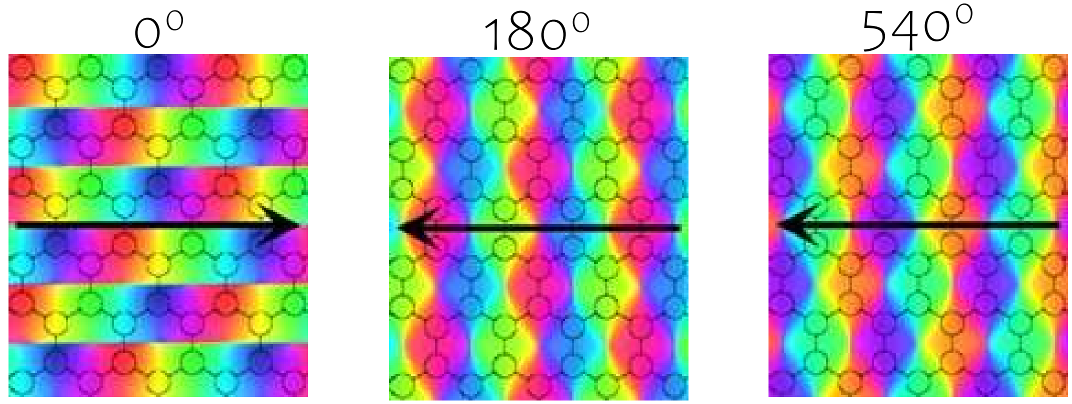
Dirac cones in the K and K' points:

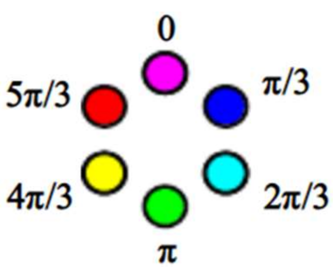


Visualizing pseudo-spin textures

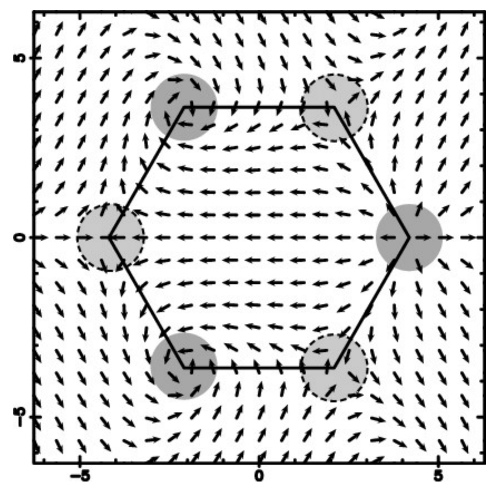
Rotating the k-vector in real space:

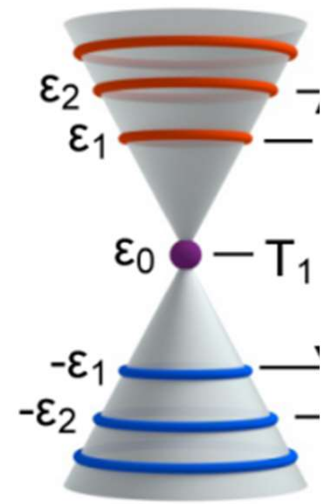






Pseudo-spin textures in k-space:

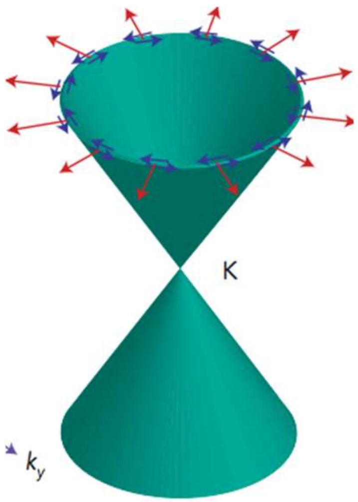


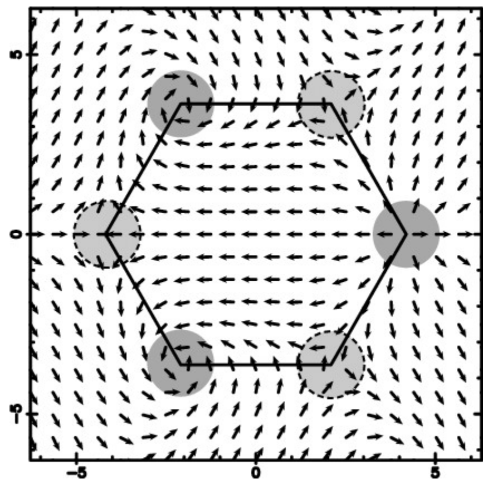


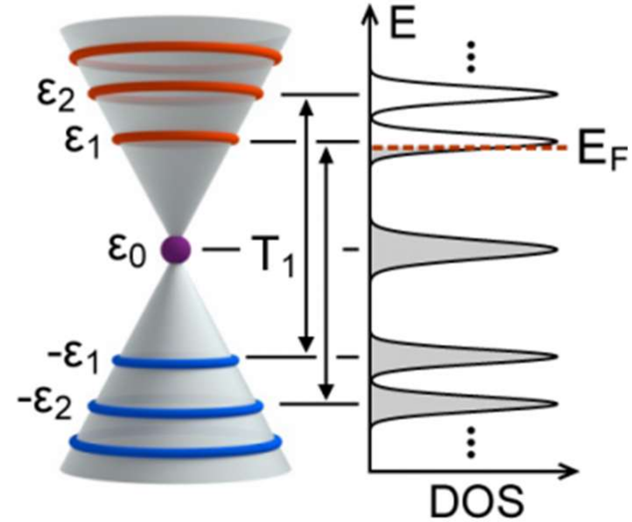
Berry's phase of π and non-trivial topological properties.

Berry curvature in graphene

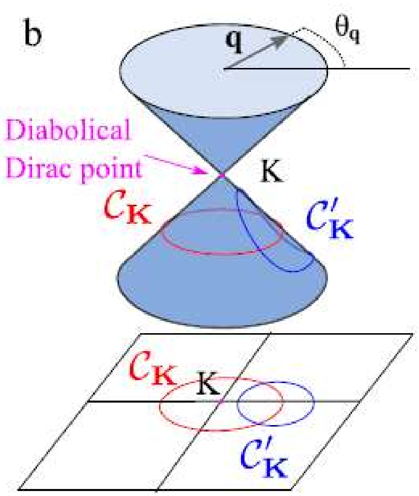
Pseudo-spin textures in k-space:

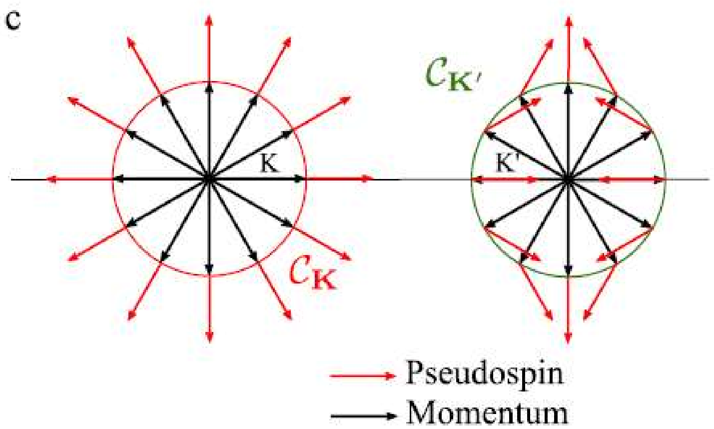






Trajectories around Dirac point in k-space:





Dirac points are Berry curvature monopoles

$$\Omega(k) = \nabla \times \mathcal{A}$$

$$C = \frac{1}{2\pi} \oint_{BZ} \Omega \, dk^2 = \nu$$